Tutorial 8

Exercise 1. *Let $\mathcal{R} \subset \mathbb{R}^2$ be a closed and bounded convex set, $(\mu, \nu) \in \mathcal{R}$ and $(\alpha, \beta) = A(\mathcal{R}, (\mu, \nu))$ be the arbitration pair with $\alpha \neq \mu$. Suppose the boundary of $\mathcal{R}$ is given, locally at (α, β), by the graph of a differentiable function $f(x)$ with $f(\alpha) = \beta$. Prove $f'(\alpha)$ equals the negative of the slope of the line joining (μ, ν) and (α, β).*

Solution. Let $g(u, v) = (u - \mu)(v - \nu)$ on $\mathcal{R}$. In the intersection of the bargaining set and a neighborbood of (α, β), we have

$$g(u, v) = (u - \mu)(f(u) - \nu) := h(u).$$

Since g attains its maximum at (α, β), we have $h'(\alpha) = 0$, which implies easily that $f'(\alpha) = -\frac{\beta - \nu}{\alpha - \mu}$.

Exercise 2. *Let $\mathcal{R} = \{(u, v) : v \geq 0,\ \text{and}\ u^2 + v \leq 4\}$. Sketch the bargaining sets and find the arbitration pairs $A(\mathcal{R}, (\mu, \nu))$ using the following points as the status quo point (μ, ν).*

(i) $(0, 0)$.

(ii) $(0, 1)$.

Solution. (i) The bargaining set is shown in Figure 2. On the bargaining set,

$$g(u, v) = (u - 0)(v - 0) = uv = u(4 - u^2).$$

Since $g'(u) = 4 - 3u^2, g'(u) = 0 \Longrightarrow u = \frac{2\sqrt{3}}{3}$. It is easy to see g attains its maximum at $u = \frac{2\sqrt{3}}{3}$ on the bargaining set. Hence in this case, we have arbitration pair $(\frac{2\sqrt{3}}{3}, \frac{8}{3})$.

(ii) When $(\mu, \nu) = (0, 1)$, the bargaining is shown in Figure 3. In this case,

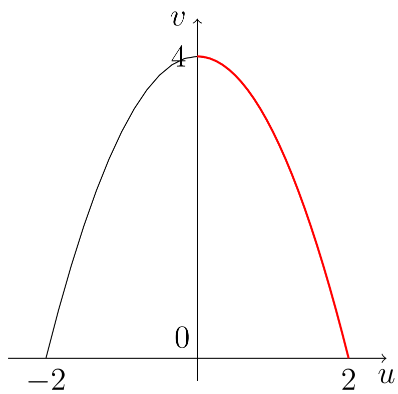

Figure 1

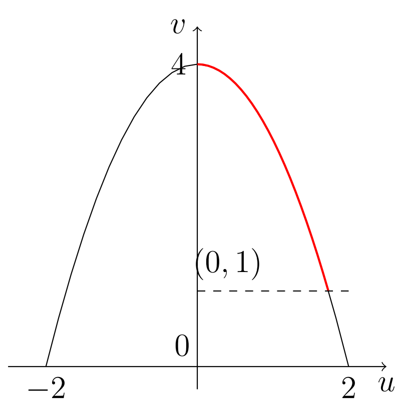

Figure 2

on the bargaining set,

$$g(u,v) = (u-0)(v-1) = u(4-u^2-1) = 3u-u^3.$$

Let $g'(u) = 0$, we get $u = 1$. The arbitration pair is $(1, 3)$.

Cooperative games

Let $\mathcal{A} = \{A_1, \cdots, A_n\}$ be the set of players. Let X_i be the set of strategies of Player A_i $(i = 1, \cdots, n)$. In a cooperative game, the players can cooperate by forming coalitions. Players in the same coalition can share their utility.

Coalition. We call any subset $S \subset \mathcal{A}$ a coalition.

If the strategies and the corresponding payoff vectors are given, we define $\nu(S)$ to be the total maximin value of S when the game is viewed as a two-person non-cooperative game between S and S^c. More precisely,

$$\nu(S) = \text{ the value of the payoff matrix of } S.$$

For convenience, we write $\nu(A) = \nu(\{A\})$. As a function on $2^{\mathcal{A}}$ (the power set of $\mathcal{A}$), ν satisfies

$$\nu(S \cup T) \geq \nu(S) + \nu(T), \text{ for any } S, T \subset \mathcal{A} \text{ with } S \cap T = \emptyset.$$

Characteristic function (form). A function $\nu : 2^{\mathcal{A}} \to \mathbb{R}^+$ is called a characteristic function (form) if $\nu(\emptyset) = 0$ and

$$\nu(S \cup T) \geq \nu(S) + \nu(T), \text{ for any } S, T \subset \mathcal{A} \text{ with } S \cap T = \emptyset.$$

Essential game: $\nu(\mathcal{A}) > \sum_{i=1}^n \nu(A_i)$.

Inessential game: $\nu(\mathcal{A}) = \sum_{i=1}^n \nu(A_i)$.

Solving a cooperative game: find reasonable ways to split the total payoff $\nu(A)$ among the players.

In this sense, only essential games are of interest, since we have

Theorem 1. *If ν is inessential, then*

$$\nu(S) = \sum_{A \in S} \nu(A), \text{ for any } S \subset \mathcal{A}.$$

Imputation.

A vector $(x_1, \cdots, x_n) \in \mathbb{R}^n$ is called an imputation for ν if

(i) $\nu(A_i) \le x_i$ for $1 \le i \le n$.

(ii) $\sum_{i=1}^n x_i = \nu(\mathcal{A})$.

Let $I(\nu)$ to be the set of imputations. We understand $I(\nu)$ as the set of all possible ways to split the total payoff.

Solution concept 1: the core.

The *core* of ν is defined by

$$C(\nu) = \{\boldsymbol{x} \in I(\nu) : \boldsymbol{x} \not\prec_S \boldsymbol{y} \text{ for any } \boldsymbol{y} \in I(\nu) \text{ and any } S \subset \mathcal{A}\}.$$

where $\boldsymbol{x} \prec_S \boldsymbol{y}$ means $x_i < y_i$ for $A_i \in S$ and $\sum_{A_i \in S} y_i \le \nu(S)$.

Characterization of $C(\nu)$.

Theorem 2. *Let $\boldsymbol{x} = (x_1, \cdots, x_n) \in I(\nu)$. Then $\boldsymbol{x} \in C(\nu)$ if and only if*

$$\nu(S) \le \sum_{A_i \in S} x_i \text{ for any } S \subset \mathcal{A}.$$

Disadvantage: $C(\nu)$ may be an empty set.

Solution concept 2: Shapley values.

For each player A_i, define the Shapley value of A_i by

$$\phi_i = \sum_{A_i \in S \subset \mathcal{A}} \frac{(n-|S|)!(|S|-1)!}{n!}(\nu(S) - \nu(S \setminus \{A_i\})).$$

Let $\boldsymbol{\phi}(v) = (\phi_1, \cdots, \phi_n)$.

ϕ_i can be viewed as the average contribution of Player A_i.

Exercise 3. *Let $\mathcal{A} = \{1, \cdots, N\}$. Verify that for each $i \in \mathcal{A}$,*

$$\sum_{i \in S \subset \mathcal{A}} \frac{(N-|S|)!(|S|-1)!}{N!} = 1.$$

Advantages of Shapley values:

(i) $\boldsymbol{\phi}(\nu) \in I(\nu)$.

(ii) $\boldsymbol{\phi}(\nu)$ is the unique payoff allocation which satisfies the axioms for Shapley value.

(iii) If the game is convex, then $\boldsymbol{\phi}(\nu) \in C(\nu)$, in particular $C(\nu) \neq \emptyset$.

Convex games. We say a game ν is convex if

$$\nu(S \cup T) \geq \nu(S) + \nu(T) - \nu(S \cap T)$$

for any $S, T \subset \mathcal{A}$.

Exercise 4. *Consider a 3-person game with player set $\mathcal{A} = \{1, 2, 3\}$ and each of the players has strategy set $\{0, 1\}$. Suppose the payoffs are given in Table 1.*

(i) Find the characteristic function ν.

(ii) Find the core and draw the region on the x_1-x_2 plane.

(iii) Find the Shapley values.

Strategies	Payoff vectors
$(0, 0, 0)$	$(2, 3, 4)$
$(0, 0, 1)$	$(4, 6, 4)$
$(0, 1, 0)$	$(7, 4, 2)$
$(0, 1, 1)$	$(3, 2, 9)$
$(1, 0, 0)$	$(4, 3, 5)$
$(1, 0, 1)$	$(5, 8, 7)$
$(1, 1, 0)$	$(3, 1, 5)$
$(1, 1, 1)$	$(3, 6, 5)$

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Table 1